An Extension to the Theory of Fluctuations in an Equilibrium Convective Ensemble

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ABSTRACT

The theory of fluctuations in an equilibrium convective ensemble emerging in the literature is revisited and extended in this study. The probability of requiring n mutually independently convective plumes and a total cloud-base mass flux M for subgrid convection to occur in a given grid box is derived based on the concept of the grand canonical ensemble, which is well known in classic statistical mechanics. The probability distribution functions of the cloud-base mass flux and the number of subgrid convective plumes are dependent on the average of each of the two quantities. This problem has been considered in previous work (e.g., Craig and Cohen 2006), where n was distributed as a Poisson process. It turns out that deriving the distribution describing n simultaneously with that describing the cloud-base mass flux yields a geometric distribution rather than a Poisson distribution. In fact, though the two distributions are quite different, they are logically consistent since the geometric distribution can result if the rate parameter connecting n with the cloud mass flux M of a Poisson distribution is itself random and distributed exponentially. Other, physically based distributions for the rate parameter are possible, and we introduce one based on a stochastic model of vertical velocity. The work here is thus an extension rather than an alternative to the Craig-Cohen theory.

1. Introduction

Error and uncertainty in numerical weather and climate prediction models are rooted, to a great degree, in the inevitable discrete approximations on a computational grid required to solve the governing equations of weather and climate. One of the most challenging problems in the discrete approximation is to account for sub-grid unresolved physical processes. Subgrid moist convection is among the most energetic of these processes, particularly in the tropics, which is treated using the parameterization approach. The term "parameterization" in numerical atmospheric models has traditionally been applied to the representation of unresolved dynamical processes within a grid box by the average values of processes resolved in that grid box. In the recent years, this term has been expanded to include stochastic representations (e.g., Buizza et al. 1999).

A very interesting stochastic model of cumulus convection ensemble was presented in a couple of landmark papers (Craig and Cohen 2006, CC1_06 hereafter; Cohen and Craig 2006), which developed a theory of equilibrium fluctuations in a field of cumulus clouds based on the Gibbs canonical ensemble of statistical mechanics. Under the assumptions of large-scale forcing and the limit of non-interactive convective cells, CC1_06 showed that the probability density function (*pdf*) of individual cloud mass fluxes is exponential and derived an expression for the total mass flux over a region of given size based on a Poisson distribution of the number of convective clouds in a grid box.

Traditionally in operational weather and climate prediction models, the effect of unresolved subgrid convection on the prediction of resolved scales is parameterized

deterministically as an ensemble mean, and the stochastic fluctuations about this ensemble mean are ignored. It has recently been advocated that the stochastic fluctuations should be properly accounted for in the subgrid parameterization in order to address a persistent issue in operational ensemble prediction: the spread of ensemble members tends to be underestimated. To this end, efforts have been taken to develop a theoretical framework to shed light on the equilibrium distribution of subgrid convection. In this study, a grand canonical ensemble approach is used to derive the probability distribution functions of n distinct convective plumes and a corresponding total cloud-base mass flux M in a given NWP grid box, given that their means are known.

2. Grand Canonical Ensemble Approaches

Classes of cumulus parameterizations for which the Grand Canonical ensemble approach is appropriate are those that replace the ensemble of clouds in a grid box by a single representative cloud (e.g., Kain and Fritsch 1993; Arakawa and Schubert 1974; Grell 1993; Pan and Wu 1974). As mentioned above, the parameters specifying that cloud have traditionally been deterministic, and modifications to this approach involve stochastic parameters depending on the *pdf*s of sub-grid scale properties. In the following, we review two approaches to the implementation of Grand Canonical ensembles to sub-grid scale parameterization.

a)   Review of Craig and Cohen (2006)

A particular insight by CC1_06 is the conceptual model of additive mass flux due to the individual clouds, visualized as a segment of individual subgrid "convective cell" of mass flux m_i adding up to a total mass flux M, whose average is $\langle M \rangle$. Assume that

the ensemble of deep clouds in a given NWP model grid box (see Fig. 1) is made of a number of distinct convective plumes. Given the total number of convective plumes n within a grid box, the corresponding total mass flux in the grid box at cloud base M is defined as

$$M = \sum_{i=1}^{n} m_i ,$$ (1)

where m_i (the mass flux at the base of individual clouds), and where M and n are randomly varying around their means, $<M>$ and $<n>$, which are assumed to be known.

CC1_06 began their analysis assuming that, for given M, the number n of convective plumes in a grid cell was Poisson distributed with a rate constant λ equal to the inverse of average mass flux $<m>$ at the bottom of a single convective plume:

$$p_M(n) = \frac{(\lambda M)^n e^{-\lambda M}}{n!}$$ (2)

They then showed that the mass flux m_i at the bottom of each convective plume was exponentially distributed. Combining the distributions for the individual plumes with the Poisson distribution for the number of plumes, Craig and Cohen derive the *pdf* of total mass flux in a grid cell as (their Eq. 14):

$$p(M) = \left(\frac{<n>}{<m>}\right)^{1/2} \exp(-<n>) M^{-1/2} \exp(-M/<m>) I_1\left(2\sqrt{\frac{<n>}{<m>}M}\right).$$ (3)

To summarize, CC1_06 first fix M and use the Poisson distribution for n to derive an exponential distribution for m. Note that with M truly constant, $<M> = M$ and λ is

5

constant. Replacing λM with $\lambda <M> = <n>$ in Eq. (2), they then use the exponential distribution of m and the Poisson distribution for $<n>$ to derive Eq. 2.

b)   Simultaneous Entropy Maximization

A different expression of $p(M)$ is obtained when the grand canonical ensemble concept from classic statistical mechanics is employed in this context using maximum entropy principles. Following Penland (1988), we assume that all combinations $\{n, m_i\}$ are equally probable subject to the constraints that the total average mass flux $<M>$ and total average number of clouds $<n>$ are known and maximize the information entropy jointly with respect to n and the mass flux of the individual plumes $\{m_i\}$. This technique essentially makes the same assumption involving non-overlapping convective clouds as in CC1_06. As in CC1_06, the *pdf* of the mass flux m_i at the bottom of each convective plume is found to be exponential:

$$p(m_i) = \frac{\langle n \rangle}{\langle M \rangle} \exp\left(-\frac{\langle n \rangle}{\langle M \rangle} m_i\right). \tag{4}$$

Eq. (4) is identical to CC1_06's Eq. (16), employing the relation $<n>/<M> = 1/<m>$, their Eq. (1).

In contrast to CC1_06, the *pdf* for n, does not obey a Poisson law, but is geometrically distributed (Penland 1988):

$$p(n) = \frac{1}{\langle n \rangle}\left(1 - \frac{1}{\langle n \rangle}\right)^{n-1}. \tag{5}$$

Thus, with $<M>$ and $<n>$ in equilibrium with large-scale forcing, the *pdf* of the total mass flux at cloud base M is found to be a simple exponential:

$$p(M) = \frac{1}{<M>} \exp\left(\frac{-M}{<M>}\right). \tag{6}$$

c)   *Reconciliation*

Let us begin with a basic assumption of CC1_06: that n obeys a Poisson distribution. How does this come about? Assume that there are N grid boxes in Fig. 1. Thus, there are $<n>N$ convective plumes in the grid. Then the probability of finding exactly n plumes in a given grid box for a large N is given by the binomial distribution:

$$B_n\left(<n>N; \frac{1}{N}\right) = \left(\begin{array}{c} <n>N \\ n \end{array}\right)\left(\frac{1}{N}\right)^n\left(1 - \frac{1}{N}\right) \tag{7}$$

As $N \rightarrow +\infty$, the above binomial distribution approaches a Poisson distribution (i.e., the probability of finding exactly n plumes in a box within the grid of an infinite number of boxes):

$$p_{<n>}(n) = \frac{e^{-<n>} <n>^n}{n!}. \tag{8}$$

This probability distribution is the same as Eq. (8) in CC1_06.

There are two ways to interpret this derivation of the Poisson distribution in terms of the grid boxes in a GCM: First, we may interpret N as being the actual number of grid boxes in that GCM and $<n>$ is the average over the N grid boxes, in which case $<n>$ is the same for all grid boxes and N is large but actually finite. Second, we may interpret N as the number of possible values for n in a single grid box, and then $<n>$ represents the

average over all those possibilities. The first interpretation is clearly unrealistic; why should the average number of convective clouds be the same over a tropical ocean and a midlatitude desert? The second interpretation is more realistic, and the one in which we interpret the results of CC1_06, although verification using statistics over the entire domain of a cloud-resolving model is thus rendered conceptually difficult.
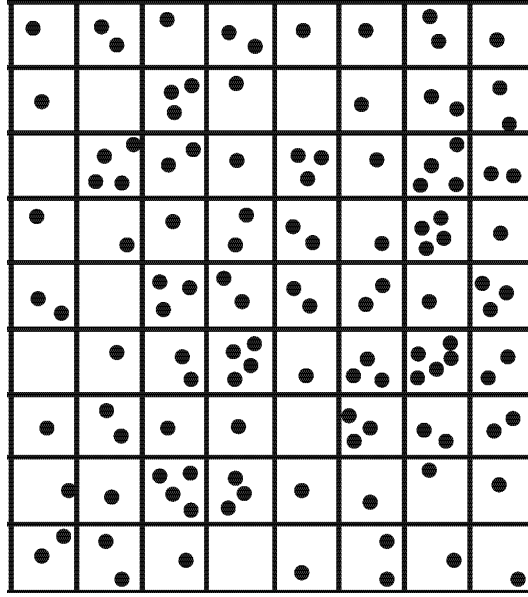

Figure 1: Illustration of a grand canonical ensemble approach in a computational domain. There are N boxes in the grid. Dots in the horizontal computational grid boxes represent distinct convective plumes. All the boxes have equally probability of triggering convection with the average number of convective plumes per box being $<n>$ and the corresponding total mass flux at cloud base being $<M>$.

Let us return to Eq. (2), which is also Eq. (2) in CC1_06. However, we recognize M as a random variable and rewrite the conditional *pdf* for n given M as

$$p_{<n|M>}(n) = \frac{(\alpha\lambda < M >)^n}{n!}\exp(-\alpha\lambda < M >). \tag{9}$$

where $\alpha = M/<M>$ and λ is constant. That is, the rate parameter connecting n and M is now a random variable $\alpha\lambda$ with average λ; α is a random variable with unit mean. The marginal distribution of a Poisson process n with a random rate parameter belongs to a family of *negative binomial distributions* when the *pdf* of that rate parameter is a Γ–distribution, of which the exponential distribution is a special case (Greenwood and Yule 1920). The geometric distribution belongs to the family of negative binomial distributions and results when the rate parameter is exponentially distributed. More generally, classes of Poisson systems with random rate parameters are known as "mixed Poisson process," "modified Poisson processes," or "Cox processes" (Kallenberg 1974; Grandell 1997; Karlis and Xekalaki 2005). Analytical expressions for the marginal distribution of n in these systems are rare due to the algebraic difficulty of performing the integral over the distribution of the rate parameter.

In terms of Fig. 1, Eq. (8) assumes that in any grid box, the number of convecting cells n in a grid box is Poisson distributed with average number $<n>$ and that the mass flux m at the base of each cell is exponentially distributed. Further, Eq. (8) implicitly assumes that the relationship between $<n>$ and $<M>$ is constant and known; although $<n>$ and $<M>$ may change from grid box to grid box, the relationship between $<n>$ and $<M>$ is proportional and described by a constant λ. Eq. (4) results from maximum entropy considerations. It may also be derived as in CC1_06, by assuming that n is Poisson distributed and using an exponential distribution for m. However, this Poisson distribution is conditioned on the total mass flux M, which is also exponentially

distributed. This is equivalent to allowing the proportional relationship between $<n>$ and $<M>$ to vary randomly from grid box to grid box, with the constant of proportionality obeying an exponential distribution. We discuss this further below.

3. Extension to a physically based rate parameter

a)   *Theoretical considerations*

As stated above, the geometric distribution is the distribution resulting from the Poisson distribution with a rate parameter that is itself a random variable obeying an exponential distribution (Greenwood and Yule 1920; Consul and Jain 1973), which is a special case of the Γ-distributions. Thus, application of Penland's (1988) result does not compete with, but rather extends, the Craig-Cohen formalism. We statistically interpret the maximum entropy result as follows: In the absence of other information, n may be assumed to obey a Poisson distribution. However, a constant rate parameter λ does not maximize the entropy, and may itself be replaced by a random variable without the introduction of any other parameters. If all that is known are the means $<n>$ and $<M>$, the entropy is maximized with the simplest distribution available to α since there is no justification for assuming anything more complicated. As we have seen, this extension is also a simplification. Using the geometric distribution for n results in a simple exponential distribution for M, so that the grid total mass flux M need not be drawn from a *pdf* involving modified Bessel functions. The simplicity of the geometric and exponential distributions allows several possible implementations in a numerical atmospheric model.

There are certainly more physically based possibilities for the distribution of the random coefficient α, and we present one here. First, let us recall that cloud formation is unlikely to occur when air is descending. Thus, it is reasonable to allow α, and thus M, to be proportional to the vertical velocity w for $w > 0$ and to set α to zero otherwise; the probability that α is zero is equal to the probability that $w < 0$. Vertical velocity is subject to the laws of hydrodynamics, so that more information than knowledge of just $<M>$ and $<n>$ is available. If the fluctuating hydrodynamic equations (e.g., Landau and Lifschitz 1959), which are generally quadratic in velocity, can be reduced to a linear stochastic differential equation with correlated additive and multiplicative (CAM) noise on the timescales of interest (e.g., Sardeshmukh and Sura 2009), then deviations w' from its mean, i.e., anomalies of w, obey a stochastic differential equation of the form

$$\frac{dw'}{dt} = Lw' + (Ew' + g)\eta_1 + b\eta_2 - \frac{1}{2}Eg, \tag{10}$$

where L, E, g and b are constants, and where η_1 and η_2 are independent Gaussian white noises. Thus, the *pdf* of w' can be evaluated and found to be a *stochastically generated skewed* (SGS) distribution of the following form:

$$p(w') = \frac{1}{\mathcal{N}}\left((Ew' + g)^2 + b^2\right)^{-(v+1)} \exp\left[\frac{2gv}{b}\arctan\left(\frac{Ew' + g}{b}\right)\right]. \tag{11}$$

In Eq. (11), $v = -[L/E^2 + 1/2]$, and $\mathcal{N}$ is a normalization constant. Note that ascent occurs even when $<w>$ is negative if $w' > -<w>$. Note also that since the mean of α is unity, it is the deviation from the mean $\alpha' = \alpha - 1$ that is proportional to w'. We therefore choose a *pdf* for α having the form of Eq. (11), but with w' replaced by $(\alpha-1)$.

It is time to step back and address an apparent inconsistency in the above paragraph. From Eq. (9), n and α are properties of the grid box as a whole. However, our justification for the CAM noise process appears to be appropriate for individual cells within the grid box. A negative value of the vertical velocity averaged over a grid box $<w>_g$ does not preclude subgrid instances of positive vertical velocity. It is thus conceivable that n is nonzero even when $<w>_g$ is negative. Of course, this is the reason for a stochastic parameterization in the first place. The grid average $<w>_g$ is the resolved vertical velocity generated by the numerical model. The random variations in α are meant to model combined effects of subgrid convection. If observed or analyzed vertical velocity averaged over the size of a grid box is distributed as Eq. (11), then a numerical parameterization of subgrid scale convection using the same functional form of Eq. (11) implies the same type of self-similarity enjoyed by Gaussian processes. That is, we must assume that the *pdf* of the combined processes, each of which is described by such an SGS distribution, is the same type of SGS distribution, just as the *pdf* of the sum of Gaussian variables is also a Gaussian variable. This type of self-similarity is not strictly true for SGS systems described by Eqs. (10-11). However, Penland and Sardeshmukh (2012) have shown that such SGS *pdf*s are very similar to *pdfs* that do have this self-similar property, and we are confident that any practical effects of making such an assumption are small. In any case, single-plume parameterizations such as the Arakawa-Schubert scheme implicitly make this assumption anyway.

Eqs. (10) and (11) enable the ability of allowing stochastic forcing of a quantity to depend on that quantity itself. Further, studies of both daily data and analyses (Sura et al. 2006; Sura and Sardeshmukh 2008) of a variety of variables at different atmospheric

heights show that Eq. (10) is often more realistic at short timescales than an equation with only additive stochastic forcing. Still, if data or theory demand Gaussianity, it is simple to set E = 0 in Eq. (10) and then Eq. (11) reduces to a Gaussian *pdf*. Stochastic differential equations similar, though not identical, to Eq. (10) have been used in the past to model atmospheric turbulence (e.g., Flesch and Wilson 1992).

The skew in the *pdf* (Eq. 11) is stochastically generated and obtains only when $g \neq 0$. Eq. (11) is greatly simplified if the additive and multiplicative noises are uncorrelated, i.e., if $g = 0$. In that case, the *pdf* simply reduces to a Student's *t*-distribution with non-integer degrees of freedom. This is consistent with the results of Plant and Craig (2008; their Fig. 1), who showed the frequency plot of total convective mass flux per unit area, as sampled from a cloud resolving model (Cohen and Craig 2006), to be remarkably symmetric about the mean. Thus, although numerical implementation of the skewed case is almost as easy as implementation as the un-skewed case, we consider the special case of un-skewed (un-CAM) noise in the rest of this section for expository purposes.

A *pdf* for *M* based on Eq. (11) is not as simple as the exponential distribution, and the corresponding marginal distribution $p(n)$ for *n* is much more complicated than either the geometric distribution or the Poisson distribution with constant rate parameter. With this model for *M*, even for $g = 0$, evaluation of $p(n)$ involves a transcendental integral:

$$p(n) = \frac{(\lambda <M>)^n}{\mathcal{N} n!} \int_0^\infty \frac{\alpha^n \exp(-\alpha \lambda <M>)}{\left[E^2 (\alpha-1)^2 + b^2 \right]^{(\nu+1)}} d\alpha \qquad n > 0$$

(12)

$$p(n=0) = 1 - \sum_{n=1}^\infty p(n) \qquad n=0$$

13

The expression for $p(n)$ has some interesting properties. First, the expression for $p(n=0)$ has no simple expression because there is a contribution to that *pdf* even when $\alpha \neq 0$, and the contribution to $p(n=0)$ when α *does* vanish is not easily evaluated because the mean of α is unity, not zero. Next, although the integrand is very well behaved, there does not seem to be a solution of the integral as a simple combination of known functions. The integral would be a gamma function if there were no denominator in the integrand, but there is. If there were no additive noise, that expression could be transformed into one involving exponential integrals. However, the condition of multiplicative but no additive noise is unphysical since that would imply that the vertical velocity would eventually collapse to its mean value and never change again. Without the exponential term, the integral would imply an upper bound on the number of convective cells, but for finite λM any value of n is permitted. Although reasonable values of E and b (cf. Eq. 19 of Sardeshmukh and Sura 2009) allow the integrand to be apparently well-approximated (correlation > 0.99) as α^n multiplied by a decaying exponential, deviations from that approximation occur near $\alpha = 0$ and for large α (in the power-law tail), resulting in an error of a factor of two when $n = 1$. Fortunately, the integrand does have some advantages: for large enough ν, the *t*-distribution is approximately Gaussian, and for any given n, the integral is easily evaluated numerically. Even better, convective parameterizations don't really need the marginal distribution of n; the quantity M is easily sampled, after which n, if needed, may be sampled from its conditional *pdf*, Eq. (9).

In practice, it will be easier to generate w', and thus M, using the stochastic differential equation (10) rather than sampling from the *pdf*. This will have the further

advantage of automatically generating an ensemble of cloud lifetimes, since the lagged covariances of w' would be exponential, with decorrelation time determined by L and E^2.

b)   *Numerical examples*

Consider the case where M depends only on the total vertical velocity $w = w'+<w_g>$. The grid size cloud mass flux M might be set to zero if w is negative. For positive w, M would be parameterized as being proportional to w. Please note that other dynamical quantities, such as divergence, may be used as proxies for M if w is not available to the model.

The Craig-Cohen *pdf*, exponential *pdf*, and a *pdf* for M based on Eq. (9) are compared in Fig. 2. In Fig. 2, parameters have been chosen consistent with Craig and Cohen (2006): $<N> = 5$, $<m> = 10^7$ kg s^{-1}, and $<M> = 5\times10^7$ kg s^{-1}. The combination of parameters L and E were chosen so that the mean decay time of the cloud ($T_d=[L+E^2/2]^{-1}$) was 45 minutes, similar to the constant cloud lifetime chosen in Plant and Craig (2008). Both the exponential and the Craig-Cohen *pdf*s are characterized by two parameters. For consistent parameters yielding the same $<M>$, the exponential has a larger $<M^2>$ than does the Craig-Cohen *pdf*. The *pdf* of the linear model was chosen in a region of mean atmospheric ascent, and the corresponding $<M^2>$ was chosen to equal that of the exponential distribution. However, unlike the exponential distribution the *pdf* of the linear model has a power-law tail so that large excursions of M away from its mean are more probable.
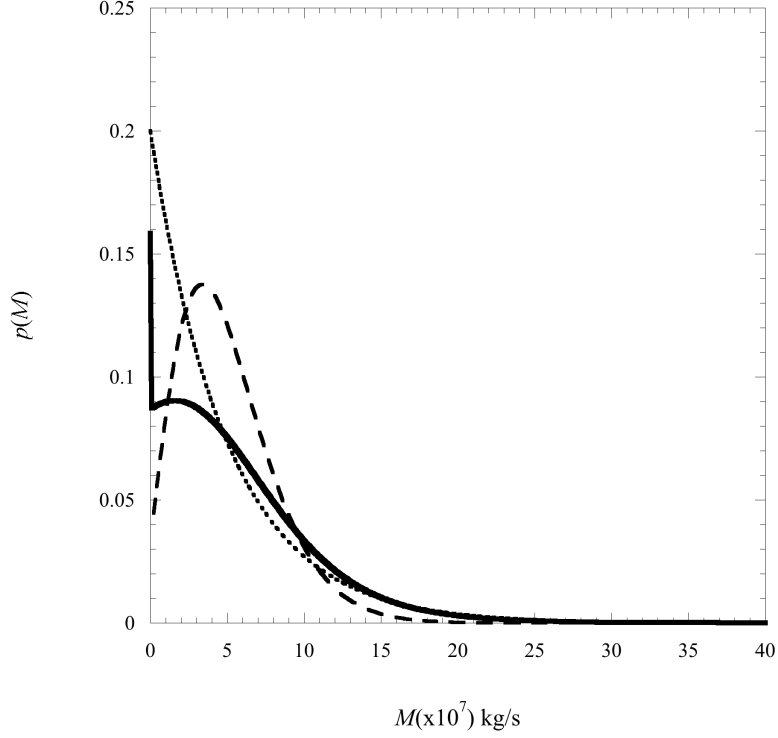
Fig. 2: Comparison of *pdf*s p(M) of grid size mass flux perturbation for <N> = 5 and <M> = 5x10^{7}kg/s.   Dashed: Craig-Cohen (Eq. 3).   Dotted: Exponential (Eq. 6).   Solid: Stochastic linear model (Eq. 11).

4. Discussion and Conclusions

The probability of requiring *n* distinct convective plumes and a total cloud-base mass flux *M* for subgrid convection to occur in a given grid box has been previously derived based on the concept of the grand canonical ensemble, which is well known in classic statistical mechanics. The probability distribution functions of the cloud-base mass flux and the number of subgrid convective plumes are dependent on the average of each of the two quantities. For a large number of such grid boxes in a given area, the concept can be extended to a homogenous stochastic situation. In this situation, the probability of finding exactly *n* subgrid convective plumes in one of the grid boxes is

16

given by the binomial distribution, which converges to the Poisson distribution when the number of the grid boxes approaches infinity.

In a series of seminal papers, Craig and Cohen (2006), Cohen and Craig (2006) and Plant and Craig (2008) developed a stochastic convection scheme based on equilibrium convection ensembles. The average number $<n>$ of plumes in a gridbox is related to the average total amount of mass $<M>$ at the bottom of the clouds within each grid box through a rate parameter λ. If λ is constant, than n is described as a Poisson process. In this paper, we suggest extensions to their work with a family of parameterizations based on modified Poisson distributions of n. We first consider a case based on maximization of entropy, reproducing the Craig-Cohen result that the mass flux at the bottom of individual clouds is exponentially distributed. However, it is found that the distribution of n is not obviously Poisson, but is rather geometrically distributed (Penland 1988). This is equivalent to the Craig-Cohen theory where λ is random and exponentially distributed, indicating no information about the physical processes governing the cloud formation within the grid box (Greenwood and Yule 1920).

The geometric distribution has been used previously in rainfall modeling (e.g., Kavvas and Delleur 1981) to describe the number of convective cells observed at a single observation site during a single storm. Models of this type (e.g., Ramirez and Bras 1985) are cluster-point models (Neyman and Scott 1952); they assume that the number of storm arrivals with time is Poisson, but that the time between convective events in a single storm is distributed exponentially so that the number of convective cells in that storm is geometrically distributed. The maximum entropy results of Penland (1988) suggest that

this conceptual scenario, as envisioned by Craig and Cohen (2006) in a more restricted manner, may be used in a spatial context.

An alternative approach to a stochastic parameterization of convection may wish to restrict the phase space from which M is sampled to values at least approximately consistent with the equations of motion, including relevant conservation laws. We suggest a stochastic parameterization where the distribution of M is determined by the distribution of vertical velocity. For descending air motion, M and n are taken to be zero in that grid box. For ascending motion, M is sampled from a distribution of the form given in Eq. (11), and n can then be sampled from the Poisson distribution conditioned on M. The procedure has several advantages. First, the distribution of M is physically based. Also, because the vertical motion is related to the large-scale field in the numerical model, we expect correlation of parameters over several grid boxes to arise naturally rather than as a product of further parameterization (e.g., Plant and Craig 2008). Further, the lifetimes of clouds need not be specified separately, but are rather a product of the parameterization scheme, which has an exponential decorrelation time (Sardeshmukh and Sura 2009) determined by the parameters of the pdfs of w'. Finally, it may be possible to simplify the parameterization even further: for a sufficient number of non-interacting clouds, the Central Limit Theorem allows the distribution of M, which is the sum of the mass flux at the bottom of the individual clouds, to be approximately Gaussian. This simplification is equivalent to setting $E = 0$ in Eqs. (10) and (11).

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